Bonus 3: Solution

1. If a is a double root of $f(x)$, then we know that there exists a polynomial $g(x)$ so that $f(x) = (x-a)^2 g(x)$.

   Then $f(a) = (0)g(a) = 0$, so a is a root of $f(x)$. Furthermore, if we take the derivative of f, we get

   $$f'(x) = (x-a)^2 g'(x) + 2(x-a)g'(x)$$

   So $f'(a) = (0)g(0) + 2(0)g'(0) = 0$, so a is also a root of $f'(x)$.

2. Since a is a root of $f(x)$, we know $f(x) = (x-a)g(x)$ for some polynomial $g(x)$. Taking the derivative, we get

   $$f'(x) = g(x) + (x-a)g'(x)$$

   But since a is also a root of $f'(x)$, we know that $f'(x) = (x-a)h(x)$ for some other polynomial $h(x)$. So we have:

   $$(x-a)h(x) = g(x) + (x-a)g'(x)$$

   If we then solve for $g(x)$, we get

   $$g(x) = (x-a)h(x) - (x-a)g'(x) = (x-a)[h(x) - g'(x)]$$

   Substituing this back into the equation $f(x) = (x-a)g(x)$, we find that

   $$f(x) = (x-a)(x-a)[h(x) - g'(x)] = (x-a)^2[h(x) - g'(x)]$$

   So indeed a is a double root of $f(x)$.